The gauge invariant quark Green's function in two-dimensional QCD

H. Sazdjian

Institut de Physique Nucléaire, CNRS/IN2P3,
Université Paris-Sud 11, F-91405 Orsay, France
E-mail: sazdjian@ipno.in2p3.fr

Abstract. The gauge invariant quark Green's function, defined with a path-ordered phase factor along a straight-line, is studied in two-dimensional QCD in the large-N_c limit by means of an exact integrodifferential equation. It is found to be infrared finite with singularities represented by an infinite number of threshold type branch points with a power of -3/2, starting at positive mass squared values. The Green's function is analytically determined.

Keywords: QCD, quark, gluon, Wilson loop, gauge invariant Green's function.
PACS: 12.38.Aw, 12.38.Lg

GAUGE INVARIANT GREEN'S FUNCTIONS

Gauge invariant Green's functions are expected to provide more reliable information about the physical properties of observables than gauge variant ones. In QCD, they are defined with the aid of path-ordered gluon field phase factors [1, 2]. We report in this talk on recent results obtained for the gauge invariant quark Green's function in two-dimensional QCD [3, 4]. Using for the paths skew-polygonal lines, it is possible to derive for the latter in any dimensions an exact integrodifferential equation in functional form; its restriction to two dimensions in the large N_c limit then allows us to solve analytically the above equation and to have an explicit check of the spectral properties of the quark fields.

For quarks, the gauge invariant two-point Green's function is defined as

$$S_{\alpha\beta}(x,x';C_{x'x}) = -\frac{1}{N_c}\,\langle\overline{\psi}_\beta(x')\,U(C_{x'x};x',x)\,\psi_\alpha(x)\rangle, \tag{1}$$

α and β being the Dirac spinor indices, while the color indices are implicitly summed; U is a path-ordered gluon field phase factor along a line $C_{x'x}$ joining a point x to a point x', with an orientation defined from x to x':

$$U(C_{x'x};x',x) = Pe^{-ig\int_x^{x'} dz^\mu A_\mu(z)}. \tag{2}$$

Green's functions with paths along skew-polygonal lines are of particular interest, since they can be decomposed into the succession of simpler straight line segments. For such lines with n sides and $n-1$ junction points y_1, y_2, ..., y_{n-1} between the segments,

we define:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = -\frac{1}{N_c}\langle\overline{\psi}(x')U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_1,x)\psi(x)\rangle, \quad (3)$$

where now each U is along a straight line segment. The simplest such function corresponds to $n=1$, for which the points x and x' are joined by a single straight line:

$$S_{(1)}(x,x') \equiv S(x,x') = -\frac{1}{N_c}\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle. \quad (4)$$

(We shall generally omit the index 1 from that function.)

INTEGRODIFFERENTIAL EQUATION

To quantize the theory one may proceed in two steps. First, one integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops [5, 6, 7, 8, 9, 10]. To achieve the latter program, we use for the quark propagator in external field a representation which involves phase factors along straight lines together with the full gauge invariant quark Green's function [3, 11]. The latter feature allows implicit summation of self-energy effects at each step of the operation. This representation is a generalization of the one introduced by Eichten and Feinberg when dealing with the heavy quark limit [12].

The quark propagator in the external gluon field is expanded around the following gauge covariant quantity:

$$\left[\widetilde{S}(x,x')\right]^a_{\ b} \equiv S(x,x')\left[U(x,x')\right]^a_{\ b}. \quad (5)$$

It is possible to set up an integral equation realizing iteratively the previous expansion. Its systematic use leads to the derivation of functional relations between the Green's functions $S_{(n)}$ (skew-polygonal line with n segments) and S (one segment).

Using then the equations of motion relative to the Green's functions, one establishes the following equation for $S(x,x')$ [3]:

$$(i\gamma.\partial_{(x)}-m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu\Big\{K_{2\mu}(x',x,y_1)\,S_{(2)}(y_1,x';x)$$
$$+\sum_{n=3}^{\infty}K_{n\mu}(x',x,y_1,\ldots,y_{n-1})\,S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2})\Big\}, \quad (6)$$

where the kernel K_n ($n=2,3,\ldots$) contains globally n derivatives of Wilson loop averages with skew-polygonal contours and also the Green's function S and its derivative. The Green's functions $S_{(n)}$ being themselves related to the simplest Green's function S through series expansions resulting from functional relations, Eq. (6) is ultimately an integrodifferential equation for S. One expects that the kernels with small number of derivatives will provide the most salient contributions. Therefore, the first kernel K_2 in Eq. (6) would contain the leading effect of the interaction.

INTEREST OF THE QUARK GREEN'S FUNCTION

The interest of the gauge invariant quark Green's function is related to its particular status. If the theory is confining, then it is not possible to cut the Green's function and to saturate it with a complete set of physical states (hadrons), which are color singlets. Intermediate states are necessarily colored states. This would suggest that the Green's function does not have any singularity. However, the equation it satisfies [Eq. (6)], derived from the QCD Lagrangian, contains singularities, generated by the free quark propagator (the inverse of the Dirac operator in the left-hand side of Eq. (6)).

The above paradoxical situation is overcome with the acceptance that quarks and gluons continue forming a complete set of states with positive energies and could be used for any saturation scheme of intermediate states. It is the resolution of the equations of motion which should indicate to us how the related singularities combine to form the complete solutions.

Therefore, the knowledge of the gauge invariant quark Green's function provides a direct information about the effect of confinement in the colored sector of quarks.

SPECTRAL FUNCTIONS

Green's functions with paths along straight lines are dependent only on the end points of the paths. The transition is then simple to momentum space by Fourier transformation.

It is advantageous to consider for that purpose the path-ordered phase factor in its representation given by the formal series expansion in terms of the coupling constant g.

Using for each term of the series, together with the quark fields, the spectral analysis with intermediate states and causality, one arrives at a generalized form of the Källén–Lehmann representation for the Green's function S in momentum space, in which the cut lies on the positive real axis staring from the quark mass squared m^2 and extending to infinity [13, 14, 15, 16, 17].

Taking into account translation invariance, we introduce the Fourier transform of the Green's function S into momentum space:

$$S(x,x') = S(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.(x-x')}\, S(p). \qquad (7)$$

$S(p)$ has the following representation in terms of real spectral functions $\rho_1^{(n)}$ and $\rho_0^{(n)}$ $(n = 1,\ldots,\infty)$:

$$S(p) = i\int_0^{\infty} ds' \sum_{n=1}^{\infty} \frac{\left[\,\gamma.p\,\rho_1^{(n)}(s') + \rho_0^{(n)}(s')\,\right]}{(p^2 - s' + i\varepsilon)^n}. \qquad (8)$$

Depending on the degrees of the singularities at threshold, simplifications may occur by integrations by parts, or otherwise by summation, reducing the series into more compact forms.

TWO-DIMENSIONAL QCD

Many simplifications occur in two-dimensional QCD at large N_c [18, 19, 20]. This theory is expected to have the essential features of confinement observed in four dimensions, with the additional simplification that asymptotic freedom is realized here in a trivial way, since the theory is superrenormalizable. For simple contours, Wilson loop averages in two dimensions are exponential functionals of the areas enclosed by the contours [21, 22, 23]. Furthermore, at large N_c, crossed diagrams and quark loop contributions disappear.

It turns out that in two dimensions and at large N_c, only the lowest-order kernel K_2 survives in Eq. (6). The equation reduces then to the following form [4]:

$$(i\gamma.\partial - m)S(x) = i\delta^2(x) - \sigma\gamma^\mu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})x^\nu x^\beta \\ \times\left[\int_0^1 d\lambda\,\lambda^2\,S((1-\lambda)x)\gamma^\alpha S(\lambda x) + \int_1^\infty d\xi\,S((1-\xi)x)\gamma^\alpha S(\xi x)\right], \qquad (9)$$

where σ is the string tension.

The equation is solved by decomposing S into Lorentz invariant parts:

$$S(p) = \gamma.pF_1(p^2) + F_0(p^2), \qquad (10)$$

or, in x-space:

$$S(x) = \frac{1}{2\pi}\left(\frac{i\gamma.x}{r}\widetilde{F}_1(r) + \widetilde{F}_0(r)\right), \qquad r = \sqrt{-x^2}. \qquad (11)$$

One obtains, with the introduction of the Lorentz invariant functions, two coupled equations. Their resolution proceeds through several steps, mainly based on the analyticity properties resulting from the spectral representation (8). The solutions are obtained in explicit form for any value of the quark mass m.

The covariant functions $F_1(p^2)$ and $F_0(p^2)$ are, for complex p^2:

$$F_1(p^2) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n\,\frac{1}{(M_n^2 - p^2)^{3/2}}, \qquad (12)$$

$$F_0(p^2) = i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty}(-1)^n b_n\,\frac{M_n}{(M_n^2 - p^2)^{3/2}}. \qquad (13)$$

The masses M_n ($n = 1, 2, \ldots$) have positive values greater than the quark mass m and are labelled with increasing values with respect to n; their squares represent the locations of branch point singularities with power $-3/2$. The masses M_n and the coefficients b_n satisfy an infinite set of coupled algebraic equations that are solved numerically. Their asymptotic behaviors for large n, such that $\sigma\pi n \gg m^2$, are:

$$M_n^2 \simeq \sigma\pi n, \qquad b_n \simeq \frac{\sigma^2}{M_n}. \qquad (14)$$

In x-space, the solutions are:

$$\widetilde{F}_1(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n\,e^{-M_n r}, \qquad \widetilde{F}_0(r) = \frac{\pi}{2\sigma}\sum_{n=1}^{\infty}(-1)^{n+1} b_n\,e^{-M_n r}. \qquad (15)$$

$[r = \sqrt{-x^2}.]$

At high energies, the solutions satisfy asymptotic freedom [24]:

$$F_1(p^2) \underset{p^2 \to -\infty}{=} \frac{i}{p^2}, \tag{16}$$

$$F_0(p^2) \underset{p^2 \to -\infty}{=} \frac{im}{p^2}, \qquad m \neq 0, \tag{17}$$

$$F_0(p^2) \underset{p^2 \to -\infty}{=} \frac{2i\sigma}{N_c} \frac{\langle \overline{\psi}\psi \rangle}{(p^2)^2}, \qquad m = 0, \tag{18}$$

where in the last equation we have introduced the one-flavor quark condensate.

We present in Fig. 1 the function iF_0 for spacelike p and in Fig. 2 its real part for timelike p, for the case $m = 0$.

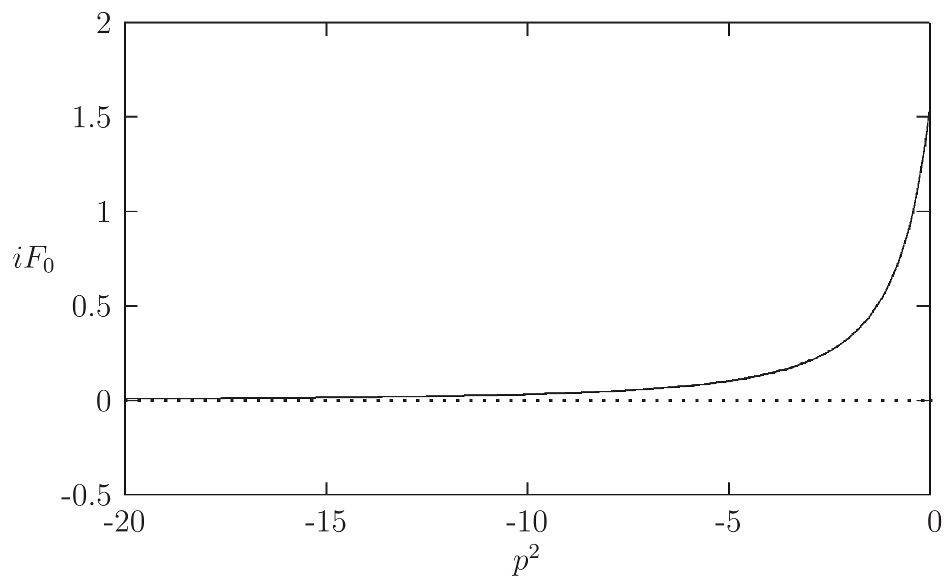

FIGURE 1. The function iF_0 for spacelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

CONCLUSION

The spectral functions of the quark Green's function are infrared finite and lie on the positive real axis of p^2. No singularities in the complex plane or on the negative real axis have been found. This means that quarks contribute like physical particles with positive energies. (In two dimensions there are no physical gluons.)

The singularities of the Green's function are represented by an infinite number of threshold type singularities, characterized by a power of $-3/2$ and positive masses M_n $(n = 1, 2, \ldots)$. The corresponding singularities are stronger than simple poles and this feature might mean difficulty in the observability of quarks as asymptotic states.

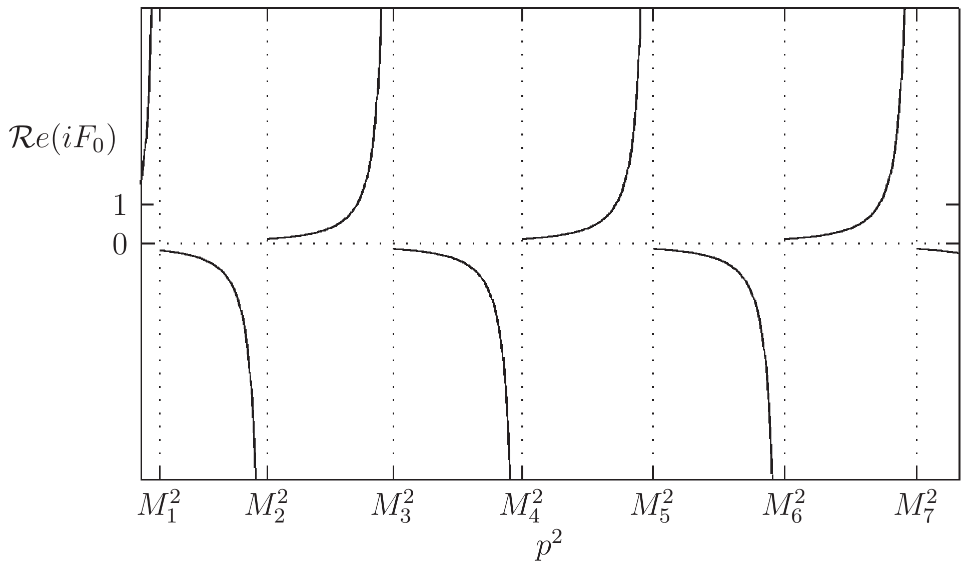

FIGURE 2. The real part of the function iF_0 for timelike p, in mass unit of $\sqrt{\sigma/\pi}$, for $m = 0$.

The threshold masses M_n represent dynamically generated masses and maintain the scalar part of the Green's function at a nonzero value even when the quark mass is zero.

ACKNOWLEDGMENTS

This work was supported in part by the EU network FLAVIANET, under Contract No. MRTN-CT-2006-035482, and by the European Community Research Infrastructure Integrating Activity "Study of Strongly Interacting Matter" (acronym HadronPhysics2, Grant Agreement No. 227431), under the Seventh Framework Programme of EU. I thank H. M. Fried and A. Pineda for stimulating discussions during the Workshop.

REFERENCES

1. S. Mandelstam, *Phys. Rev.* **175**, 1580 (1968).
2. Y. Nambu, *Phys. Lett. B* **80**, 372 (1979).
3. H. Sazdjian, *Phys. Rev. D* **77**, 045028 (2008).
4. H. Sazdjian, *Phys. Rev. D* **81**, 114008 (2010).
5. K. Wilson, *Phys. Rev. D* **10**, 2445 (1974).
6. A. M. Polyakov, *Nucl. Phys. B* **164**, 171 (1980).
7. Y. M. Makeenko, and A. A. Migdal, *Phys. Lett. B* **88**, 135 (1979).
8. Y. M. Makeenko, and A. A. Migdal, *Nucl. Phys. B* **188**, 269 (1981).
9. A. A. Migdal, *Phys. Rep.* **102**, 199 (1983).
10. Y. Makeenko, *arXiv* **hep-th/0001047** (2000).
11. F. Jugeau, and H. Sazdjian, *Nucl. Phys. B* **670**, 221 (2003).

12. E. Eichten, and F. Feinberg, *Phys. Rev. D* **23**, 2724 (1981).
13. G. Källén, *Helv. Phys. Acta* **25**, 417 (1952).
14. H. Lehmann, *Nuovo Cimento* **11**, 342 (1954).
15. A. S. Wightman, *Phys. Rev.* **101**, 860 (1956).
16. S. S. Schweber, *An introduction to relativistic quantum field theory*, Row, Peterson and Co., Evanston, 1961, pp. 721–742.
17. G. 't Hooft, and M. Veltman, Diagrammar, Tech. Rep. 73-9, CERN (1973).
18. G. 't Hooft, *Nucl. Phys. B* **72**, 461 (1974).
19. G. 't Hooft, *Nucl. Phys. B* **75**, 461 (1974).
20. C. G. Callan, Jr., N. Coote, and D. J. Gross, *Phys. Rev. D* **13**, 1649 (1976).
21. V. A. Kazakov, and I. K. Kostov, *Nucl. Phys. B* **176**, 199 (1980).
22. V. A. Kazakov, *Nucl. Phys. B* **179**, 283 (1981).
23. N. E. Bralić, *Phys. Rev. D* **22**, 3090 (1980).
24. H. D. Politzer, *Nucl. Phys. B* **117**, 397 (1976).